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LETTER TO THE EDITOR

Non-bijective canonical transformations in quantum mechanics

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Abstract. We propose a new formulation for the representation of non-bijective canonical transformations in quantum mechanics. These are represented by a unitary operator acting on a Hilbert space $\mathcal{H} \otimes \mathcal{H}$ similar to the one used for the description of a two-dimensional system.

The representation of canonical transformations in quantum mechanics has been extensively discussed by Moshinsky and his collaborators (Mello and Moshinsky 1975, Kramer *et al* 1978, Moshinsky and Seligman 1978, 1979a, b, Garcia-Calderon and Moshinsky 1980, Deenen *et al* 1980). One of the most interesting applications of this theory is to solve complicated problems. Suppose, indeed, we have a canonical transformation that maps one Hamiltonian onto a much simpler one; then all known properties of the simple problem can be translated to the other. If the spectra of both Hamiltonians are the same, the canonical transformation is bijective. When the spectra are different, the transformation, if it exists, becomes non-bijective. Recently such a transformation has been discussed (Moshinsky and Seligman 1980) in connection with collective states in nuclei.

To find the quantum representation of non-bijective canonical transformations, we are faced with two kinds of difficulties. The first one is related to the quantisation process. In order to find the quantum operator associated with a given classical function in phase space we shall use Weyl's quantisation process.

In this Letter we want to discuss a second type of difficulty arising from the non-bijectiveness of the canonical transformation. In general, the classical canonical transformation

$$\bar{q} = f(q, p) \qquad \bar{p} = g(q, p) \qquad \text{with } \{\bar{q}, \bar{p}\} = 1 \tag{1a}$$

is written in quantum mechanics as

$$\bar{Q} = F(Q, P) \qquad \bar{P} = G(Q, P). \tag{1b}$$

If $F(Q, P)$ and $G(Q, P)$ have the same spectrum as Q and P and $[F, G] = \mathrm{i}$, it is possible to find a unitary operator U such that

$$\bar{Q} = UQU^+ \qquad \text{and} \qquad \bar{P} = UPU^+.$$

This is the case for the linear canonical transformation (Moshinsky and Quesne 1971) and for some nonlinear transformations (Mello and Moshinsky 1975). In our case the operator U does not exist because Q and $\bar{Q}$ (P and $\bar{P}$) have not the same spectrum. One attempt to overcome this difficulty has been given by Plebansky and Seligman (1978). They propose to embed the Hilbert space $\mathscr{H}$ in a larger one and replace the unitarity by the concept of partial isometry. Unfortunately this embedding of the space depends strongly on the canonical transformation: for each non-bijective canonical transformation there is in general one specific embedding of the space $\mathscr{H}$. This means that it is impossible to combine canonical transformations. For example, the canonical transformation defined by the implicit equations

$$\tfrac{1}{2}(\bar{p}^2+\bar{q}^2)=p^2+q^2 \qquad 2\tan^{-1}(\bar{p}/\bar{q})=\tan^{-1}(p/q) \tag{2}$$

is represented by a set of partial isometries between $\mathscr{H}$ and $\mathscr{H}\otimes V_2$ where V_2 is a two-dimensional space related to the ambiguity spin (Moshinsky and Seligman 1978).

We want to show here that it is possible to associate with the canonical transformation (2) a unitary operator acting on $\mathscr{H}\otimes\mathscr{H}$, the second space taking the place of the V_2 space considered by Plebansky and Seligman. In other words, we intend to prove that we can make the canonical transformation bijective by considering it as a two-dimensional transformation.

We first introduce the usual creation and annihilation operators of the harmonic oscillator (HO)

$$a^+=(1/\sqrt{2})(Q-\mathrm{i}P) \qquad a=(1/\sqrt{2})(Q+\mathrm{i}P)$$

and the number operator $N=a^+a$. With these operators, the first equation of (2) can be written as

$$\bar{N}=2N \tag{3}$$

and we see clearly that a unitary operator that maps N onto $2N$ does not exist. Nevertheless there exists a family of partial isometries (PI) that map N onto the set of two operators $2N+\sigma I$ ($\sigma=0,1$). These are

$$T_\sigma=\sum_n |n\rangle\langle 2n+\sigma| \tag{4}$$

where $|n\rangle$ denotes the eigenvector of N associated with the eigenvalue n. The set of operators (4) is a complete set of partial isometries. We have indeed

$$\begin{aligned} &T_\sigma T^+_{\sigma'}=\delta_{\sigma\sigma'}I \\ &T^+_\sigma T_\sigma=\mathscr{P}_\sigma \qquad \text{is a projector} \\ &\sum_\sigma \mathscr{P}_\sigma=I. \end{aligned} \tag{5}$$

If we look at the action of these PI on the number operator N, we see that they map N onto the set $(2N, 2N+I)$:

$$\begin{aligned} T_\sigma N T^+_\sigma &=\sum |n\rangle\langle 2n+\sigma|N|2n'+\sigma\rangle\langle n'| \\ &=\sum |n\rangle(2n+\sigma)\langle n| \\ &=2N+\sigma I. \end{aligned}$$

In particular, T_0 maps N onto $2N$ but it is not unitary as can be seen from (5).

In the following we shall put a tilde on an operator if it acts on the second $\mathscr{H}$-space of $\mathscr{H}\otimes\mathscr{H}$; otherwise it will act on the first space which is the physical one. The second $\mathscr{H}$ is an auxiliary space, playing the same role as V_2 in the Plebansky–Seligman formulation.

From the PI defined above we can construct a unitary operator in $\mathcal{H}\otimes\mathcal{H}$:

$$\mathbb{U}=\sum_{\sigma} T_\sigma \tilde{T}_\sigma^+. \tag{6}$$

To prove the unitarity of this operator, we use the relations (5) and the fact that the T_σ and $\tilde{T}_\sigma$ commute as they act on different spaces:

$$\mathbb{U}\mathbb{U}^+=\sum T_\sigma \tilde{T}_\sigma^+ \tilde{T}_{\sigma'} T_{\sigma'}^+ = \mathbb{I}$$

and

$$\mathbb{U}^+\mathbb{U}=\mathbb{I}$$

where $\mathbb{I}$ denotes the unit operator in $\mathcal{H}\otimes\mathcal{H}$.

Let us show how the number operator N is transformed by $\mathbb{U}$. We have

$$\begin{aligned}\mathbb{U}N\mathbb{U}^+ &= \sum T_\sigma \tilde{T}_\sigma^+ N \tilde{T}_{\sigma'} T_{\sigma'}^+ \\ &= \sum (2N+\sigma I)\tilde{\mathscr{P}}_\sigma \\ &= 2N+\tilde{S}\end{aligned}$$

where we omit the unit operators I and $\tilde{I}$ and define $\tilde{S}=\Sigma\,\sigma\tilde{\mathscr{P}}_\sigma=\tilde{\mathscr{P}}_1$. Thus N is mapped onto $2N$ by $\mathbb{U}$ if we add an operator $\tilde{S}$ with eigenvalues 0 and 1, which acts on an auxiliary space.

The result is formally the same as that obtained by Kramer *et al*, but as they use a two-dimensional auxiliary space, $\tilde{S}$ is represented by a 2×2 matrix $\left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$. In our case $\tilde{S}$ can be represented by an infinite block diagonal matrix built with this 2×2 matrix. In both formulations, the role of $\tilde{S}$ is very important because it ensures that N and $2N+\tilde{S}$ have the same spectrum. This is a necessary condition for the existence of a unitary operator.

We can go further in the analogy between these two formulations. Let us look for example at the transformation of creation and annihilation operators a^+ and a.

Using the definition (6) of $\mathbb{U}$ and the properties of the partial isometries (5), we obtain

$$\mathbb{U}a\mathbb{U}^+=\sqrt{2}a\tilde{J}_+ + (2N+1)^{1/2}\tilde{J}_- \qquad \mathbb{U}a^+\mathbb{U}^+=\sqrt{2}a^+\tilde{J}_- + (2N+1)^{1/2}\tilde{J}_+ \tag{7}$$

where we introduce a spin operator $\tilde{\boldsymbol{J}}$ in the auxiliary space:

$$\tilde{J}_0=\tfrac{1}{2}\sum\left(|\widetilde{2n+1}\rangle\langle\widetilde{2n+1}| - |\widetilde{2n}\rangle\langle\widetilde{2n}|\right)$$
$$\tilde{J}_+=\sum|\widetilde{2n+1}\rangle\langle\widetilde{2n}| \qquad \tilde{J}_-=\sum|\widetilde{2n}\rangle\langle\widetilde{2n+1}|.$$

The expressions (7) are similar, in our formulation, to the formulae (5.8) and (5.9) of Kramer *et al* (1978).

Finally, let us sketch what happens for a more complicated canonical transformation, namely

$$|\bar{q}|=\tfrac{1}{2}(p^2+q^2) \qquad (|\bar{q}|/\bar{q})\bar{p}=\tan^{-1}(p/q).$$

In this case, the quantum picture leads us to consider a mapping between the position operator Q and the HO operator N. If we denote by $|q)$ the eigenstates of Q and by $|n\rangle$ the eigenstates of N, it is easy to show that the set

$$T_{s\lambda}=\sum|n\rangle(s(n+\lambda)| \qquad \text{where } s=\pm 1,\ \lambda\in[0,1)$$

form a complete set of partial isometries in the sense of (5). Moreover, the PI $T_{s\lambda}$ maps Q onto $s(N+\lambda)$:

$$T_{s\lambda}QT^{+}_{s\lambda}=s(N+\lambda)$$

and the unitary operator $\mathbb{U}=\Sigma_s\int \mathrm{d}\lambda\, T_{s\lambda}\tilde{T}^{+}_{s\lambda}$ represents the canonical transformation. The final result

$$\mathbb{U}|Q|\mathbb{U}^{+}=N+\tilde{\Lambda},$$

where $\tilde{\Lambda}=\Sigma\int \mathrm{d}\lambda\, \lambda\tilde{T}^{+}_{s\lambda}\tilde{T}_{s\lambda}$, is in agreement with what has been done before for this special canonical transformation (Moshinsky and Seligman 1978). This formulation can be generalised to any non-bijective canonical transformation provided we have a family of partial isometries. The auxiliary space introduced here is in general too large (a two-dimensional space V_2 is sufficient for the first example), but has the advantage that the representation space is the same, namely $\mathscr{H}\otimes\mathscr{H}$ for all the canonical transformations. This allows us to combine them and thus opens the way to a quantum representation of the general symplectic group.

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References

Deenen J, Moshinsky M and Seligman T H 1980 *Ann. Phys., NY* **127** 458
Garcia-Calderon G and Moshinsky M 1980 *J. Phys. A: Math. Gen.* **13** L185
Kramer P, Moshinsky M and Seligman T H 1978 *J. Math. Phys.* **19** 683
Mello P A and Moshinsky M 1975 *J. Math. Phys.* **16** 2017
Moshinsky M and Quesne C 1971 *J. Math. Phys.* **12** 1772
Moshinsky M and Seligman T H 1978 *Ann. Phys., NY* **114** 243
—— 1979a *Ann. Phys., NY* **120** 402
—— 1979b *J. Phys. A: Math. Gen.* **12** L135
—— 1980 to be published
Plebansky J F and Seligman T H 1978 to be published